Q.P. Code – 50823

Third Year B.Sc. Degree Examination

OCTOBER/NOVEMBER 2014

(Directorate of Distance Education)

(DSC 230) Paper III – MATHEMATICS

Time : 3 Hours] *[Max. Marks : 90*

Instructions to Candidates :

*Answer any **SIX** full questions of the following choosing atleast **ONE** from each Part.*

PART – A

1. (a) (i) If G is a group and H is a subgroup of index 2 in G, show that H is a normal subgroup of G. **2**

(ii) If H is a normal subgroup of finite order, then $O\left(\frac{G}{H}\right) = \frac{O(G)}{O(H)}$. **2**

(b) Let M and N be two normal subgroups of a group G such that $M \cap N = \{e\}$, where 'e' is the identity element of G. Show that for every $m \in M,\ n \in N,\ mn = nm$. **5**

(c) The set $\frac{G}{H}$ of all cosets of a normal subgroup H of the group 'G' is a group under the binary operation defined by $H_a H_b = H_{ab}\ \forall\ H_a, H_b \in \frac{G}{H}$. **6**

2. (a) (i) In a ring $(R, +, \cdot)\ \forall\ a, b \in R$, show that $a(-b) = (-a)b = -(ab)$. **2**

(ii) Prove that cancellation law holds in an integral domain. **2**

(b) If U and V are the ideals of a ring R and UV be the set of all elements that can be written as finite sum of the form 'uv' where $u \in U$ and $v \in V$. Prove that 'UV' is an ideal of R. **5**

(c) If p is an integer then prove that 'pz' is a maximal ideal of $(z, +, x)$ iff p is prime. **6**

PART – B

3. (a) (i) Find the sum and product of the polynomial $f(x) = 2 + 4x + x^2$ and $g(x) = 1 + x^2$ over z_4. **2**

(ii) State division algorithm of a polynomial over a field. **2**

(b) Find GCD of the polynomial $f(x) = x^{33} - 1$ and $g(x) = x^{18} - 1$ on z and express $d(x) = a(x)f(x) + b(x)g(x)$. **5**

(c) If $N(\alpha) = a^2 - 3b^2$ where $\alpha = a + b\sqrt{3} \in z\sqrt{3}$, prove that $N(\alpha\beta) = N(\alpha) \cdot N(\beta)$ for any $\alpha, \beta \in z\sqrt{3}$. Deduce that '$\alpha$' is a unit iff $N(\alpha) = \pm 1$. **6**

4. (a) (i) Let $(z, +)$ be the group of integers and $G' = \langle 3n/n \in z \rangle$ be the group w.r.t. multiplication. Define $f : z \to y'$ by $f(x) = 3^x \ \forall \ x \in z$, then prove that f is a homomorphism. **2**

(ii) Prove that homomorphic image of an abelian group is abelian. **2**

(b) Prove that the intersection of any two ideal of a ring is again an ideal of the ring and the union of two ideals of a ring R need not be an ideal of R. **5**

(c) State and prove fundamental theorem of homomorphism of rings. **6**

PART – C

5. (a) (i) Show that, in any vector space V, $C(\alpha - \beta) = C\alpha - C\beta \quad \forall C \in F$, $\alpha, \beta \in V$. **2**

(ii) Show that, $w = \{(x_1, x_2, x_3)/x_1 + x_2 + x_3 = 0\}$ is a subspace of $V_3(R)$. **2**

(b) If w_1 and w_2 are two subspaces, then prove that their linear sum $w_1 + w_2 = \{\alpha + \beta/\alpha \in w_1 \text{ and } \beta \in w_2\}$ is also a subspace. **5**

(c) Express the polynomial $V = 5x^2 + 9x + 5$ as linear combination of the polynomials $V_1 = 4x^2 + x + 2$, $V_2 = 3x^2 - x + 1$ and $V_3 = 5x^2 + 2x + 3$. **6**

6. (a) (i) Prove that the vectors $(1, 0, 2)$, $(-1, 0, 1)$, $(0, 1, 2)$ in $V_3(R)$ are linearly independent. **2**

(ii) Show that the set containing a single non-zero vector is linearly independent. **2**

(b) Find the basis and dimension of the subspace spanned by $S = \{(1, 2, 3), (3, 1, 0), (-2, 1, 3)\}$ in $V_3(R)$. **5**

(c) Prove that if $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a set of vectors, then 'S' is linearly independent iff $\exists \ \alpha_K \in F$, such that α_K is a linear combination of its preceding vectors. **6**

PART – D

7. (a) (i) If $T : V \to W$ is a linear transformation, then prove that $T(-\alpha) = -T(\alpha) \ \forall \ \alpha \in V$. **2**

(ii) Find the matrix of the linear transformation $T : V_2(R) \to V_3(R)$ defined by $T(x, \ y) = (x + y, \ x, \ 3x - y)$. **2**

(b) Find the matrix of the linear transformation $T : V_3(R) \to V_2(R)$ defined by $T(x, y, z) = (x + y, \ y + z)$ with respect to the bases,

$B_1 = \{(-1, 0, 2), (1, 2, 3), (1, -1, -1)\}$ and $B_2 = \{(1, 2), (-2, 1)\}$. **5**

(c) Show that the linear transformation $T : V_2(R) \to V_2(R)$ defined by $T(x_1, x_2) = (ax_1, bx_2)$ is non-singular and find its inverse. **6**

8. (a) (i) Find $f_x, \ f_y$ for $f(x, y) = \sin\left(\frac{x}{y}\right)$. **2**

(ii) If $u = \frac{xy}{x + y}$ show that $x u_x + y u_y = u$. **2**

(b) If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$, show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$. **5**

(c) Show that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube. **6**
