

III Year B.Sc. Degree Examination, November 2008
 MATHEMATICS (Paper – III)
 (Directorate of Distance Education Course)

Time : 3 Hours

Max. Marks : 90

Note : Answer any SIX full questions from the following.

PART – A

1. a) i) Prove that every quotient group of an abelian group is abelian. 2
 ii) If $f : (\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$ is a mapping defined by $f(x) = x^2, \forall x \in \mathbb{R}$, find whether f is a homomorphism or not. 2
 b) If $f : G \rightarrow G^1$ is a homomorphism of groups then prove that the kerf is a normal subgroup of G . 5
 c) Prove that the mapping $f : G \rightarrow G$, where G is a group, defined by $f(a) = a^{-1}, \forall a \in G$ is an automorphism iff G is abelian. 6
2. a) i) Is Z_7 a field? why? 2
 ii) Prove that every subgroup of $(\mathbb{Z}, +, \cdot)$ is a subring of the ring $(\mathbb{Z}, +, \cdot)$ of integers. 2
 b) Prove that the union of two ideals of a ring R is an ideal of R iff one is contained in the other. 5
 c) Prove that every homomorphic image of a ring R is isomorphic to the quotient ring. 6
3. a) i) Prove that $9 + 4\sqrt{5}$ is a unit in $\mathbb{Z}(\sqrt{5})$. 2
 ii) Show that the polynomial $x^2 + 1$ is reducible over the field of complex numbers. 2
 b) Find the g.c.d of $f(x) = 4x^3 - 12x^2 - 15x - 4$ and $g(x) = 12x^2 - 24x - 15$ over $\mathbb{Q}(x)$ and express it as a linear combination of $f(x)$ and $g(x)$. 5
 c) Test the polynomial $5x^3 + 8x^2 + 6x - 4$ for rational root. 6



4. a) i) Prove that the intersection of two normal subgroups of a group is a normal subgroup of G . 2
- ii) Is every integral domain always a field? Give an example. 2
- b) Prove that a normal subgroup N of a group G is a normal subgroup G iff the product of any two right cosets of N in G is again a right coset of N in G . 5
- c) If $x^2 = x, \forall x \in R$, where R is a ring, then prove that R is commutative. 6

PART - B

5. a) i) In any vector space V , prove that $0\alpha = 0$. 2
- ii) Show that a subset $S = \{(x_1, x_2, x_3) \mid 4x_1 + 7x_2 = 0\}$ is a subspace of $V_3(R)$. 2
- b) Prove that a non empty subset S of a vector space over the field F is a subspace of V iff $C_1\alpha + C_2\beta \in S, \forall \alpha, \beta \in S$ and $C_1, C_2 \in F$. 5
- c) Construct an addition table for $Z_2(Z_2)$ and list its subspaces. 6
6. a) i) Prove that the vectors $(3, 2), (2, 5, 3)$ and $(6, 16, 10)$ are linearly dependent in $V_3(Q)$. 2
- ii) Prove that a set containing a zero vector is a linearly dependent set. 2
- b) If β is not in the subspace S but, is in the subspace spanned by S and α , then prove that α is in the subspace spanned by S and β . 5
- c) If n vectors span a vector space V over the field F and r vectors of V are linearly independent then prove that $n \geq r$. 6
7. a) i) Verify whether the transformation
 $T: V_1(R) \rightarrow V_3(R)$ defined by
 $T(x) = (x, 2x^2, x^3)$ is linear or not 2
- ii) Find the matrix of the linear transformation
 $T: V_3(R) \rightarrow V_2(R)$ defined by $T(X, Y, Z) = (X + Y, Y + Z)$ w.r.t. standard bases. 2
- b) If $T: V \rightarrow W$ is a non-singular linear map, then prove that $T^{-1}: W \rightarrow V$ is linear and bijective. 5



4. a) i) Prove that the intersection of two normal subgroups of a group is a normal subgroup of G . 2
- ii) Is every integral domain always a field? Give an example. 2
- b) Prove that a normal subgroup N of a group G is a normal subgroup G iff the product of any two right cosets of N in G is again a right coset of N in G . 5
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- c) Construct an addition table for Z_2 and list its subspaces. 6
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